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ABSTRACT

Recent research has provided a reasonably coherent picture of how children learn to add and subtract. There is clear evidence that children do not learn simply by mastering a procedure and storing in memory. Instead, learning is structured in meaningful ways, connected to previous knowledge, and adapted to new contexts. This view of learning has important implications for the way arithmetic should be taught in the first or second year of school. This paper discusses: (1) the "part whole" concept (which refers to the idea that a number can be interpreted as a whole and two parts; (2) strategies children invent to add and subtract; (3) addition strategies (direct modeling, counting strategies, and number fact strategies); (4) activities to develop direct modeling, counting, and number fact addition strategies; (5) subtraction strategies (direct modeling strategies, counting strategies, and derived number facts); (6) activities to develop modeling, counting, and number fact subtraction strategies; (7) incorrect inventions and errors; and (8) drill and practice. (JN)

STRATEGIES AND ACTIVITIES

LEARNING TO ADD AND SUBTRACT

Learning activities and implications from recent cognitive research

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PREFACE

A great deal of research, much of it supported by the National Institute of Education through R&D Centers and individual grants, has been focused on early arithmetic learning. The purpose of this booklet is to draw implications from the results of research related to learning to add and subtract.

The material is aimed at teachers of children aged 4 to 8. The booklet presents a discussion of how children learn, together with specific examples of learning activities suggested by the research. Many different methods of presenting the activities are possible. The aim is not to propose a particular curriculum or teaching method. Instead, the goal is to provide clear, concrete information to teachers about effective ways to help children learn early arithmetic concepts.

While the material is intended for early childhood teachers, many parents will find the material readable and the activities easy to do with children at home. Also, the booklet may be useful to mathematics supervisors or university educators who present inservice workshops for teachers. Much of the material will also be relevant to special educators, who work with elementary school students.

The booklet was prepared by Dr. Gerald Kulm, National Institute of Education Senior Associate in Learning and Development. The material was reviewed by Dr. Patricia Campbell, Assistant Professor of Mathematics Education, University of Maryland; Mrs. Carolyn Johnson, Elementary Teacher, West Lafayette, Indiana; and Dr. Karen Schultz, Associate Professor of Mathematics Education, Georgia State University.

CONTENTS

LEARNING TO ADD AND SUBTRACT

For many children, learning to add and subtract is their first computational task in school. A child's success or failure with the early concepts of addition and subtraction often has a lasting effect on mathematics learning opportunities in the future.

Recent research has provided a reasonably coherent picture of how children learn to add and subtract. There is clear evidence that children do not learn simply by mastering a procedure and storing in memory. Instead, learning is structured in meaningful ways, connected to previous knowledge, and adapted to new contexts. This view of learning has important implications for the way arithmetic should be taught in the first or second year of school.

PART WHOLE CONCEPT

The Part-Whole concept is one of the most important developments in early arithmetic learning. The Part-Whole concept refers to the idea that a number can be interpreted as a whole and two parts. The child forms an understanding that a number triple such as 2-5-7 is stable; that is, 7 is always the whole, 2 and 5 are always the parts. The relationship holds, no matter whether the problem is 2+5=?, 7 5=?, 2+?=7, or any other combination. This enriched understanding permits children to establish links among numbers and interpret problem situations.

INVENTED STRATEGIES

There is considerable evidence that children invent procedures for adding and subtracting. This finding is probably not surprising to observant teachers who often report unusual approaches that students use to do arithmetic. Some of the early inventions use counting strategies and seem to be based on an understanding of the Part-Whole concept. Many children invent and use these strategies efficiently, sometimes preferring them to the procedures taught in school.

Invented strategies enable children to solve problems earlier than expected in some curricula. On the other hand, children persist with counting strategies longer than many teachers realize. They are able to perform satisfactorily in arithmetic tasks without finding it necessary to memorize addition and subtraction facts. Research has shown, for example, that less than half of second graders have mastered the number facts less than 10.

The reason that children invent and use their own strategies is that they are more meaningful than memorized facts. Because invented strategies are so common, it is important to understand their nature and development. If formal arithmetic learning can be linked with children's intuition and natural inventions, learning can proceed meaningfully.

ADDITION STRATEGIES

Children follow a progression in strategies, leading to automatic recall of addition facts. Most instruction tends to move too quickly from the use of objects and counting to rote memorization. Research shows that children spend considerable time using counting strategies before mastering addition facts. Children's use of different strategies is highly variable. They often use primitive or more sophisticated strategies on the same type of problems. While the following strategies represent a progression of development, they are not stages through which all children go through.

1. Direct Modeling

- *Count all* The child represents both parts with objects, and counts the total number in the two sets. For example, to find 2+3, the child puts out groups of 2 and 3, then counts: "one, two, (pause), three, four, five."

  Children develop in the Counting All procedure from touching each object while counting, to counting as they look at the objects, to counting after the objects have been removed. If objects are not available, many children use fingers, tap a pencil, or make some other movement to represent the numbers.

2. Counting Strategies

- *Count on from first* The child starts with the first number in the problem, then counts on the second number of units. For example, to find 3+4, the child counts: "three, (pause), four, five, six, seven."

- *Count on from larger* The child starts with the larger of the two numbers in the problem, then counts on the smaller. For example, to find 2+5, the child counts: "five, (pause), six, seven."

Most children quickly learn that this is a more efficient counting strategy without being taught to use it. On the other hand, they continue to use Count On From First sometimes.

3. *Number Fact Strategies*

It is impossible to list all of the strategies in this category. Some of them combine known facts with counting; others use properties of numbers. Sometimes, these strategies are taught, but children often discover them earlier than many teachers realize. The most common Number Fact strategies are summarized here.

- ***Commutative property*** The child discovers that the order of the parts is irrelevant; the sum is the same. For example, if the child has learned 7+2=9 by Counting On, s/he also knows 2+7=9.

- ***Near doubles*** Most children learn the "doubles" addition facts first (1+1=2, 2+2=4, 3+3=6, etc). Many other facts can be derived as "near doubles" by counting on or backwards from a double. For example, to find 3+5, the child remembers that 3+3=6, then counts on 2 more to get 8.
- ***Compensation*** This strategy is also based on doubles. To find 4+6, for example, the child remembers that 5+5=10, then reasons that 4 is 1 less and 6 is one more than 5, so the sum is 10.

These strategies make use of addition facts that have already been learned, and illustrate how the later facts can be learned with meaning. These strategies are useful as children learn facts with larger numbers which are difficult to count. Research has shown that children rely less and less on counting as they remember more facts. Strategies which link counting to facts already learned provide a way to extend the child's learning in a meaningful way.

ACTIVITIES TO DEVELOP ADDITION STRATEGIES

The teacher can provide a setting which will help and encourage children to use their intuition and strategies effectively. Research shows that by the end of third grade, most children have learned their basic facts. Children should be given the opportunity to learn and use addition strategies in grades one and two. These approaches help students to memorize the facts more meaningfully and they provide a "back-up system" if a fact is forgotten.

Research shows that children in grades one and two can solve word problems if they are allowed to model the action or use counting strategies. Solving word problems provides a good setting for developing an understanding of addition, as well as for acquiring problem solving skills.

1. Developing Modeling Strategies for Addition

There are two types of addition problems for children to model: Join and Combine. In Join problems, one set is given, and another set is added. In Combine problems, two sets are given and their sum is found. Children should practice modeling both types of addition.

- *Use familiar objects* Make two sets; ask how many in each set; ask how many altogether. Let the child touch or move each object as it is counted, if necessary.

  Make one set: ask how many; add another set: ask how many altogether.

- *Use story situations* Make up problems involving money, pets, and other familiar settings. Let the child model the problem with objects, then count the total. Make up stories that reflect both Join and Combine situations.

- Use blocks or chips as counters to represent problems and stories.

- Make an "In-Out Machine" from a shoe box. Put two sets of objects into the machine: ask how many will come out.

2. Developing Counting Strategies for Addition

Counting proceeds from overt touching of objects to rapid mental activity. For some children, a gradual development may include temporary use of fingers, movements, or counting aloud to aid in keeping track of the numbers.

Once again, both addition problem types should continue to be practiced. Join problems are more suggestive of the Count On Strategy.

- *Use concrete objects* Show two sets of objects, then cover them up, reminding the child how many are in each set. Ask for the sum. Uncover and count objects to check.

  Show one set of objects and cover the other set. Find the sum, then uncover to check. Cover the larger set, then the smaller set; ask the child which is easier.

Find sums without objects, but have them available as a back-up or to check the answer, if necessary or if the child makes an error.

- *Use join and combine story problems* Start with problems that involve familiar, concrete items. Ask children to imagine the objects and find the answer. After the problem is solved, write the addition sentence.

- *Use in-out machine stories* Have children imagine putting objects into the machine, then find the sum.

3. *Developing Number Fact Strategies for Addition*

Children begin early to remember easy addition facts. Many students naturally invent strategies to use the facts they already know to solve new problems. These strategies should be encouraged and taught to other students.

- *Learn the "one-more" facts first* 2+1, 3+1, 4+1, etc. Use concrete objects if necessary, to start. Some children will understand the general idea with larger numbers; for example, 40+1, 100+1.

- *Learn the "doubles" early* 2+2, 3+3, 4+4, 5+5, etc. Use objects or games that involve doubles, pictures of symmetric objects, such as 3 branches on each side of a tree.

- *Work on near doubles as a group of related facts* For example, learn 4+5, 5+4, as facts related to the double 4+4. Stress the idea of a double and one more.

SUBTRACTION STRATEGIES

The same levels of abstraction exist for subtraction as for addition, concrete manipulation, counting strategies, and number fact strategies. With subtraction, however, the strategy used is more closely related to the type of problem being solved. In order to understand the strategies, it is necessary to identify these problem types.

There are four types of subtraction problems:

1. *Separating* Jim had 8 cookies. He gave 3 to Julie. How many cookies

did Jim have left?

2. *Partition* Martha has 9 pencils. Three are red, the rest are yellow. How many yellow pencils does she have?

3. *Joining* Kathy has 5 pennies. How many more pennies does she need, so she will have 9 pennies altogether?

4. *Comparison* Joe has 3 ballons. Jill has 7 balloons. How many more balloons does Jill have than Joe?

Only the "Separating" type of problem involves a subtractive action, usually described by teachers and students as "take-away." If the only concept related to subtraction is take-away, students will have difficulty in recognizing the other problems as subtraction when they begin to write symbolic statements later.

Research shows that children can solve the other types of subtraction problems if they are allowed to use strategies that correspond to the structure of the problem. For example, on Comparison problems, children may match two sets of objects, then count how many objects are left over in the larger set. The question of whether to add or subtract to solve the problem would not make sense to the child at this time, because a matching rather than a subtractive operation is used.

There are three distinct levels of subtraction strategies. As with addition, the strategies develop from direct modeling of the problem with concrete objects to counting, and finally, to memorized number facts.

1. Direct Modeling Strategies

- *Separating from* The child makes the larger set, then takes away or separates the smaller set. The answer is the number of objects left. This strategy is used most often on Separating and Partition problems; sometimes on Comparison problems.

- *Separating to* The child makes the larger set, then takes objects away until the smaller set is left. The number of objects removed is the answer. This strategy is often used on Separate problems when the number removed is unknown.

- *Adding on* The child makes a set equal to the smaller number, then adds on objects until the set is equal to the larger number. The answer is the number of objects added. This strategy is used almost entirely on Joining problems.

- *Matching* The child makes two sets, then matches them one-to-one. The unmatched objects are counted to give the answer. This strategy is used almost exclusively on Comparison problems.

2. Counting Strategies

- *Counting down* The child starts with the larger number and counts backwards as many as the smaller number. The number reached is the answer. Very few children use this strategy, unless the number subtracted is small, in Separate problems.

- *Counting up* The child starts counting with the smaller number, and ends with the larger number. The quantity of numbers counted is the answer. This strategy is used on Joining problems, and sometimes on Comparison problems.

- *Choice* Some children discover that Counting Down is most efficient for some problems, while Counting Up is better for others. For example, to find 8-6 it is quicker to count up from 6 to 8, but to find 8-2, it is faster to count down 2 from 8.

3. Derived Number Facts

Most derived subtraction strategies are based upon addition facts.

- *Inverses* The child recalls the inverse addition fact. For example, to find 13-7, the fact 7+6=13 is recalled, to give the answer 6.

- *Compensation* The child recalls an addition doubles fact, then adjusts it. For example, to find 17-7, recall that 8+8=16; 7 is 1 less than 8, so the answer is 9, which is 1 more than 8.

ACTIVITIES TO DEVELOP SUBTRACTION STRATEGIES

As with addition, the goal is that children memorize and have quick recall of subtraction facts. On the other hand, a firm understanding of subtraction will help future development of more complex procedures. Memorization and recall must be built upon meaningful understanding.

At first, children's subtraction skills depend on direct modeling of the action of the problem. Gradually, these procedures are replaced by more efficient, abstract processes. Research shows that understanding of subtraction develops in two ways Children become less dependent on direct models and more able to use counting or number facts. They also become more flexible in choosing a subtraction strategy, becoming less dependent on the action of the problems.

The following activities can help children develop subtraction skills. Remember that development proceeds unevenly, so a child may use abstract procedures on one type of problem, and direct modeling on another. Stable use of numerical subtraction facts takes place over an extended period of time.

1. Developing Modeling Strategies for Subtraction

Modeling strategies provide a link with the child's intuition. It is important to have materials available for modeling, even after children have stopped depending on them.

Subtraction modeling strategies are similar to those for addition, so the concepts can be developed together. All four types of subtraction problems should be used in story form with familiar settings. These stories, contrary to some opinions, are the most natural, easy way for children to understand subtraction.

- *Act out the story* Have children perform the story, using real materials to carry out the operations on the numbers. Later, substitute blocks or other objects to represent the actual items in the story.

- *Use a variety of objects* Work from using the actual items in a story to using objects to represent items. For comparison problems, use related objects, such as balls and bats or flowers and pots.

- *Use the in-out machine* Five blocks go In, 3 come Out. If 7 go In, how many will come Out?

- *Cover the objects* Let the child represent the problem, then cover the objects. Ask the child to finish the problem without the objects, then uncover and check.

2. Developing Counting Strategies for Subtraction

The shift from Direct Modeling to Counting strategies is an indication of a child's growth of understanding. At first, counting strategies for subtraction closely parallel the action of the problem.

Subtraction counting strategies are very similar to those for addition, so the concepts can be developed together. Once children are able to Count On, they can begin subtraction along with addition. This helps them to recognize contexts and choose the correct operation in a problem.

The only type of subtraction problem that cannot be solved by counting is the Comparison problem Careful instruction is needed to support the development from Direct Modeling to Number Facts for Comparison problems.

- *Use small numbers* When objects are first removed use 1, 2 or 3 as the smaller number in Separate problems, and as the difference in Join and Partition problems. Keep objects available for checking or as a "back-up" if the child has difficulty.

- *Allow counting aids* Let children use fingers, taps of a pencil, or pointing at imaginary objects. The transition from counting objects to strictly mental arithmetic proceeds through various motor and visualization stages for many children.

- *Count out loud* Counting procedures require keeping track of how many numbers have been counted. The sounds of the numbers being counted aloud can help the child with this memory task. Later, children can count silently and remember the numbers.

- *Do mental arithmetic* Start with numbers differing by one or two. Find 8-2 (Count Down); find 7-5 (Count Up). Work towards larger numbers; find 29-2 (Count Down); find 36-33 (Count Up).

3. *Developing Number Fact Strategies for Subtraction*

The development of subtraction facts is related to an understanding of the relationship between addition and subtraction. Many derived subtraction facts are based on a recall of addition facts. Children should be encouraged to derive and remember subtraction facts, but they snould be able to use counting or modeling as a back-up, especially in problem solving settings.

- *Emphasize part-part-whole* Use dominoes to illustrate parts; given any two numbers in a P-P-W combination, find the third; write related addition and subtraction sentences.

- *Learn the one-less facts* 9-1, 8-1, 7-1, etc. Later, work with larger numbers; 50-1, 99-1, 1000-1.

- *Equalize two sets* Give two children different odd numbers of objects; ask one to give some to the other so they each will have the same number of objects. Is the Whole still the same?

- *Do comparison problems* Mary has 13 cents, Bill has 5 cents. How much more does Mary have? Which number is the Part; which is the Whole?

INCORRECT INVENTIONS AND ERRORS

Children do invent incorrect procedures. One child, when asked to find the sum 4+4, said "seven." He reasoned that if 3+3=6, then 4+4=7. It is important for the teacher to be an alert observer of children's strategies and to notice the reasons for their errors. Usually, these errors are not random, but reflect an incomplete or misapplied rule.

Some errors in later work come about from rules or shortcuts which no longer apply. For example, sometimes children invent or are told the rule "always subtract the smaller number from the larger." Later, they make errors such as:

$$\begin{array}{r} 72 \\ \underline{-38} \\ 46 \end{array}$$

Instead of learning to borrow, the student subtracts from the larger in each column. Another common error involves the number zero. Students are sometimes told, when doing problems like 7-0, that "Zero is nothing, so you get the same number you started with." Later, they make errors such as:

$$\begin{array}{r} 206 \\ \underline{-\ 74} \\ 272 \end{array}$$

Once errors such as these are learned and practiced, they are sometimes difficult to change. It is important to monitor closely the errors that children make as they learn new procedures. Often, errors arise when students are placed under pressure to do a procedure they are not yet sure of, causing them to use a rule inappropriately. Many errors are based on over-generalizations. The teacher should give many different examples and types of problems so that the student can test a rule on all possible cases.

DRILL AND PRACTICE

Children continue to use modeling or counting strategies, even after they have learned many number facts. A memorized fact today may require a counting strategy tommorrow, especially in a new problem situation or under the stress of testing. Progress toward confident, consistent use of memorized facts and procedures is gradual.

While drill and practice are necessary in consolidating addition and subtraction skills, drill should not be intense with heavy emphasis on speed until children are confident. Drill given too early may cause children to fall back upon inefficient strategies, producing frustration due to time pressures. Children enjoy practicing skills with which they are successful. Games, flashcards, and other small group or individual activities can provide effective and motivational drill.

RESOURCES

Children's Arithmetic: The Learning Process by Herbert Ginsburg gives insights about how children learn mathematics. Specific teaching suggestions are made, placing a major emphasis on counting as a basis for arithmetic. Published by D. Van Nostrand Company, 1977.

Mathematics Learning in Early Childhood is a resource book for teaching mathematics to children aged 3-8. Teaching procedures and hundreds of activities emphasize problem solving and relating mathematics to the real world of the child. Available from the National Council of Teachers of Mathematics.

Developing Computational Skills provides a fresh view of teaching basic facts, algorithms, and mental arithmetic. The book also includes ideas on teaching learning disabled children and on teaching arithmetic with a calculator. Available from the National Council of Teachers and Mathematics.

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